# Every Tantrix Discovery Challenge Has a Solution

Taku Koyahata

**Abstract**

We prove that for every challenge number $n \geq 3$ and every loop colour required by the rules, the Tantrix™ Discovery puzzle has a solution: the tiles $1, \ldots, n$ (numbered cyclically modulo 10) can be placed on the hexagonal grid so that the lines of the designated colour form a single closed loop through all tiles, all touching edges match, and no empty cell is enclosed. The proof combines explicit computer-generated solutions for $n \leq 35$ with an insertion construction based on a ten-tile periodic gadget, generalising the "4-tiles' equation" of C. Fraser, which covers only the red loops with $n \equiv 4 \pmod{10}$. Since the sources disagree on the colour attached to tile 7, we treat both readings. All certificates can be checked by a short independent program.

## 1 Introduction

Tantrix™ is a tile-laying game invented by Mike McManaway in New Zealand in 1988. Its tiles are hexagons, each carrying three lines in three of the four colours red, yellow, blue and green; every line joins the midpoints of two sides. Besides the two-player game, several solitaire puzzles are sold. The simplest of them, the *Discovery* puzzle [9], uses ten tiles numbered 1 to 10, all coloured red, yellow and blue. The number on the back of each tile is printed in one of these colours. Puzzle $n$ ($3 \leq n \leq 10$) asks the player to arrange the tiles $1, \ldots, n$ so that the lines of the colour of the number $n$ form one closed loop through all $n$ tiles, while all touching sides have matching colours.

The puzzle continues beyond ten with several Discovery sets: for a *challenge number* $n$ one takes the tiles $1, 2, \ldots, n$, where the numbering restarts at 1 after every tenth tile, and the loop colour is determined by the last digit of $n$ [4]. Tantrix players keep records of the largest challenge numbers solved. A solution must not contain holes, and for an official record the arrangement must moreover be "round" [4].

It is natural to ask whether *every* challenge has a solution. Computer searches answer this for small $n$: Kino and Uno [4, 5] model the puzzle as an integer program and solve challenges up to $n = 50$. For infinitely many $n$ the only known argument is the "4-tiles' equation" of C. Fraser, reported in [4], which gives red loops for all $n = 10x + 4$; it is the reason for the roundness rule. On the complexity side, Holzer and Holzer [3] and Baumeister and Rothe [1] show that the related rotation puzzles, where the positions of the tiles are fixed and only rotations are chosen, are NP-complete.

**Contributions.**

- We prove that every challenge $n \geq 3$ has a solution in every required colour (Theorem 8). For $n \equiv 7 \pmod{10}$ we cover both red and blue loops, so the result does not depend on how the colour of tile 7 is read.
- The proof rests on a general insertion theorem (Theorem 7): a ten-tile *gadget* that is periodic in one direction can be inserted into a suitable *base* solution any number of times. The no-holes condition is handled by an Euler-characteristic argument (Lemmas 2 and 3).

- The required finite objects (three gadgets, thirteen bases and 42 small solutions) were found with the CP-SAT solver [7]. They are published together with a short independent checker (Section 7).
- As a by-product, the CP-SAT model finds solutions directly for challenges up to about $n = 70$ within minutes.

## 2 Definitions

**The grid.** We use axial coordinates for the hexagonal grid: the cells are the points of $\mathbb{Z}^2$, and the six directions are

$$D_0 = (1, 0),\ D_1 = (0, 1),\ D_2 = (-1, 1),\ D_3 = (-1, 0),\ D_4 = (0, -1),\ D_5 = (1, -1),$$

indexed by $d \in \mathbb{Z}_6$, so that $D_{d+3} = -D_d$. Two cells $p, q$ are *adjacent* if $q - p = D_d$ for some $d$; edge $d$ of cell $p$ is the side shared with $p + D_d$. We write $R(P)$ for the union of the closed hexagons of a set $P$ of cells. Two hexagons intersect if and only if their cells are adjacent (they share a side), three hexagons have a common point if and only if their cells are pairwise adjacent (they share a vertex), and no four hexagons have a common point.

**Tiles.** The Discovery set consists of ten tiles. Tile $t \in \{1, \ldots, 10\}$ is described by the colours $T_t(0), \ldots, T_t(5) \in \{\mathrm{r}, \mathrm{y}, \mathrm{b}\}$ of its edges in clockwise order (Table 1). Each colour occurs on exactly two edges of each tile, and the two edges of the same colour are joined by a line of that colour. The data are taken from Scherphuis's description of the Tantrix tiles [8], using its "new" numbering, which is the numbering of the Discovery set.

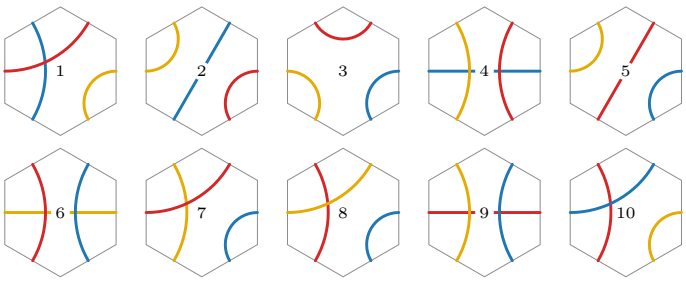



Figure 1: The ten Discovery tiles in rotation 0. Edge 0 points to the right and the edges are numbered clockwise.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| yybrbr | rrbyyb | bbyyrr | brybyr | bbryyr | ybryrb | bbyryr | bbryry | rbyryb | yyrbrb |

Table 1: Edge colours $T_t(0) \cdots T_t(5)$ of the Discovery tiles.

**Placements.** A *placement* is a finite set $P \subset \mathbb{Z}^2$ together with a tile type $\tau(p) \in \{1, \ldots, 10\}$ and a rotation $\rho(p) \in \mathbb{Z}_6$ for every $p \in P$. The colour of edge $d$ of an occupied cell $p$ is $\kappa(p, d) = T_{\tau(p)}(d - \rho(p))$.

**Challenges.** For $n \geq 3$, the challenge $n$ uses the tiles $1, 2, \ldots, n$, where the $k$-th tile has type $((k-1) \bmod 10) + 1$; thus type $t$ is used $m_t(n) = \lfloor (n-t)/10 \rfloor + 1$ times if $t \leq n$ and 0 times otherwise. The *designated colours* of $n$ depend on $n$ mod 10:

| $n$ mod 10 | 0 | 1, 2, 3, 9 | 4, 5, 7 | 6, 8 |
|---|---|---|---|---|
| colours | r, y, b | y | r | b |

For $3 \leq n \leq 10$ this is the table of the official puzzle sheet [9], which lists one loop colour for each $n \leq 9$ and states that the 10-tile puzzle can be solved in all three colours. For $n \geq 11$ the colour is determined by the last digit of $n$ [4], and we follow the same table. The sources disagree on residue 7: the official sheet gives a red loop for 7 tiles, whereas [8] lists the number of tile 7 as blue. We therefore also prove the result for blue loops with $n \equiv 7 \pmod{10}$ (Theorem 8).

**Definition 1** (Solution)**.** Let $n \geq 3$ and let $c$ be a designated colour of $n$. A placement $P$ is a *solution* of $(n, c)$ if

(S1) $|P| = n$ and exactly $m_t(n)$ cells carry tile type $t$, for every $t$;

(S2) (matching) $\kappa(p, d) = \kappa(p + D_d, d + 3)$ whenever $p, p + D_d \in P$;

(S3) (single loop) $p + D_d \in P$ whenever $p \in P$ and $\kappa(p, d) = c$, and the *loop graph* $L_c(P)$, whose edges are the pairs $\{p, p + D_d\}$ with $\kappa(p, d) = c$, is a single cycle through all cells of $P$;

(S4) (no holes) every bounded connected component of $\mathbb{R}^2 \setminus R(P)$ is empty, i.e. every empty cell is joined to the outside of a large disk by a path of adjacent empty cells.

By (S2) the relation in (S3) is symmetric, and each cell has exactly two edges of colour $c$, so $L_c(P)$ is always a disjoint union of cycles when the first part of (S3) holds.

## 3 Basic properties

For a placement $P$ let $V(P) = |P|$, let $E(P)$ be the number of unordered pairs of adjacent cells of $P$, and let $F(P)$ be the number of triples of pairwise adjacent cells of $P$. Put $\chi(P) = V(P) - E(P) + F(P)$.

**Lemma 2** (Euler characteristic)**.** *$\chi(P)$ is the Euler characteristic of $R(P)$. Consequently, if the cells of $P$ are connected under adjacency, then $P$ has no holes if and only if $\chi(P) = 1$.*

*Proof.* The closed hexagons form a finite cover of $R(P)$ by contractible sets whose nonempty intersections are contractible (a side or a vertex), so by the nerve theorem $R(P)$ is homotopy equivalent to the nerve of the cover. By the intersection properties of hexagons noted in Section 2, the nerve has $V(P)$ vertices, $E(P)$ edges and $F(P)$ triangles, hence $\chi(R(P)) = \chi(P)$. If $R(P)$ is connected, it is a compact connected planar polygonal region, and $\chi(R(P)) = 1 - h$ where $h$ is the number of bounded components of the complement. Each such component is a union of open hexagons and sides of empty cells, so it contains an empty cell that cannot reach the outside; conversely an enclosed empty cell lies in a bounded component. □

**Lemma 3** (Cut formula)**.** *Let $a \in \mathbb{Z}$ and split $P$ into $P_{<a} = \{p \in P : p_x < a\}$ and $P_{\geq a} = \{p \in P : p_x \geq a\}$. Then*

$$\chi(P) = \chi(P_{<a}) + \chi(P_{\geq a}) - \big(E_a(P) - F_a(P)\big),$$

*where $E_a(P)$ and $F_a(P)$ count the adjacent pairs and pairwise adjacent triples of $P$ that meet both $P_{<a}$ and $P_{\geq a}$. These pairs and triples consist of cells with $x$-coordinate $a - 1$ or $a$ only.*

*Proof.* Every cell, pair and triple lies entirely in $P_{<a}$, entirely in $P_{\geq a}$, or meets both. Adjacent cells differ by at most 1 in the $x$-coordinate, so a crossing pair or triple uses only the columns $a-1$ and $a$. □

Some formulations of the puzzle use, instead of (S4), a *local* gap-free rule: around every tile the occupied neighbours, read cyclically, form a single run. The next lemma shows that (S4) is the stronger of the two, so our solutions satisfy both. (The converse fails: the 18 cells at distance 1 or 2 from a fixed cell satisfy the local rule but enclose that cell.)

**Lemma 4** (No holes implies locally gap-free)**.** *Let $P$ satisfy (S3) and (S4) for some colour $c$, and let $p \in P$. Then the occupied neighbours $p + D_d$ of $p$ form a single cyclic run of consecutive directions $d$.*

*Proof.* Write $q_d = p + D_d$. If all six neighbours are occupied there is nothing to prove, so let $k \geq 1$ be the number of maximal cyclic runs of occupied neighbours, and let $a$ be the number of occupied neighbours. Since $D_{d+1} - D_d$ is again a direction, $q_d$ and $q_{d+1}$ are adjacent, while $q_d$ and $q_{d+2}$ are not. Hence the pairwise adjacent triples containing $p$ are exactly $\{p, q_d, q_{d+1}\}$ with both neighbours occupied, and there are $a-k$ of them (a run of length $\ell$ contains $\ell-1$ consecutive pairs). Removing $p$ deletes one vertex, $a$ pairs and $a-k$ triples, so

$$\chi(P \setminus \{p\}) = \chi(P) - 1 + a - (a-k) = \chi(P) - 1 + k = k,$$

using $\chi(P) = 1$ (Lemma 2; $P$ is connected by (S3)). By (S3), $L_c(P)$ is a cycle through all cells, so removing $p$ leaves a path through all cells of $P \setminus \{p\}$; thus $P \setminus \{p\}$ is connected. As in the proof of Lemma 2, a connected placement has $\chi = 1 - h \leq 1$, where $h$ is its number of holes. Therefore $k \leq 1$. □

## 4 Gadgets and bases

Fix integers $W \geq 3$ and $H \geq 1$, put $V = (W, 0)$, and let $S = \{(x, y) \in \mathbb{Z}^2 : 0 \leq y < H\}$ be the horizontal strip of height $H$. For a placement $G$ with cells in $[0, W) \times [0, H)$ let $\widehat{G} = \bigcup_{j \in \mathbb{Z}} (G + jV)$ be its periodic extension, where the tile of $p + jV$ is that of $p$. Notions such as matching and the loop graph extend verbatim to the infinite placement $\widehat{G}$.

**Definition 5** (Gadget)**.** A *gadget* of colour $c$ is a placement $G \subset [0, W) \times [0, H)$ such that

(G0) $|G| = 10$ and every tile type $1, \ldots, 10$ occurs exactly once;

(G1) $\widehat{G}$ satisfies the matching condition (S2);

(G2) $p + D_d \in \widehat{G}$ whenever $p \in \widehat{G}$ and $\kappa(p, d) = c$; the loop graph $L_c(\widehat{G})$ has exactly two connected components $\Lambda_1$ and $\Lambda_2$; and exactly two of its edges join a cell in column $W-1$ to a cell in column $W$, one in $\Lambda_1$ and one in $\Lambda_2$;

(G3) $\chi_{\mathrm{per}}(G) := \chi(G) - \big(E_W(\widehat{G}) - F_W(\widehat{G})\big) = 0$.

By periodicity, (G2) gives exactly one edge of $\Lambda_1$ and one edge of $\Lambda_2$ between columns $jW - 1$ and $jW$ for every $j \in \mathbb{Z}$, and the same counts $E_{jW}(\widehat{G}) = E_W(\widehat{G})$, $F_{jW}(\widehat{G}) = F_W(\widehat{G})$ hold for every $j$. Equivalently, $G$ describes a placement on the cylinder $(\mathbb{Z}/W\mathbb{Z}) \times [0, H)$ on which the lines of colour $c$ form two cycles, each crossing the seam once, and $\chi_{\mathrm{per}}(G)$ is the Euler characteristic of that cylindrical placement. Since $\Lambda_1$ and $\Lambda_2$ each cross every column cut exactly once, the part of $\Lambda_1$ (and of $\Lambda_2$) between two consecutive crossings lies within one period $[jW, (j+1)W) \times [0, H)$, and every cell of $\widehat{G}$ lies on $\Lambda_1$ or $\Lambda_2$.

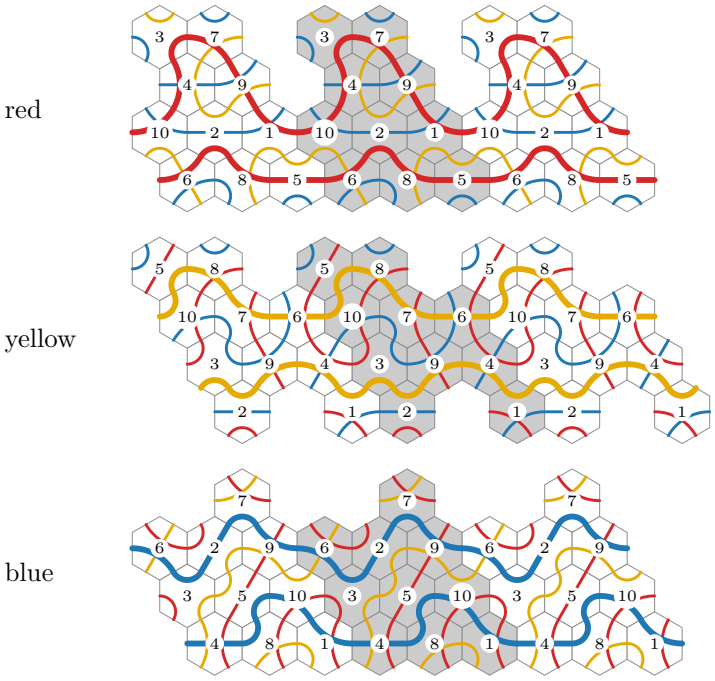



Figure 2: The gadgets $G_{\mathrm{r}}$, $G_{\mathrm{y}}$, $G_{\mathrm{b}}$ ($W = 3$, $H = 4$), shown over three periods of $\widehat{G}$; one period is shaded. The thick lines are the two strands $\Lambda_1$, $\Lambda_2$ of the loop colour.

**Definition 6** (Base)**.** Let $G$ be a gadget of colour $c$. A *base* for $G$ is a solution $B$ of $(n_0, c)$ such that $B \subset S$ and $B \cap \big([0, 2W) \times \mathbb{Z}\big) = G \cup (G + V)$, with the tiles of $\widehat{G}$.

For $k \geq 0$ define the placement

$$B_k = \{\, p : p \in B,\ p_x < W \,\} \ \cup \bigcup_{j=1}^{k} (G + jV) \ \cup\ \{\, p + kV : p \in B,\ p_x \geq W \,\},$$

where every cell keeps its tile; in words, $k$ copies of the gadget are inserted between $G$ and $G + V$, and everything to the right is shifted by $kV$.

Note that $B_0 = B$ and $B_k \cap \big([0, (k+2)W) \times \mathbb{Z}\big) = \widehat{G} \cap \big([0, (k+2)W) \times \mathbb{Z}\big)$.

## 5 The insertion theorem

**Theorem 7** (Insertion)**.** *If $B$ is a base for a gadget $G$ of colour $c$ and $B$ is a solution of $(n_0, c)$, then $B_k$ is a solution of $(n_0 + 10k, c)$ for every $k \geq 0$.*

*Proof.* Consider the three column ranges

$$U_1 = \{x < W\}, \qquad U_2 = \{0 \leq x < (k+2)W\}, \qquad U_3 = \{x \geq (k+1)W\}.$$

By construction, $B_k$ agrees with $B$ on $U_1$, with $\widehat{G}$ on $U_2$, and with the translate $B + kV$ of $B \cap \{x \geq W\}$ on $U_3$; outside the strip $S$ all three are empty.

*Locality.* Any two adjacent cells, and any three pairwise adjacent cells, lie in a common $U_i$. Indeed their $x$-coordinates differ by at most one; if they are not all $< W$ and not all $\geq (k+1)W$,

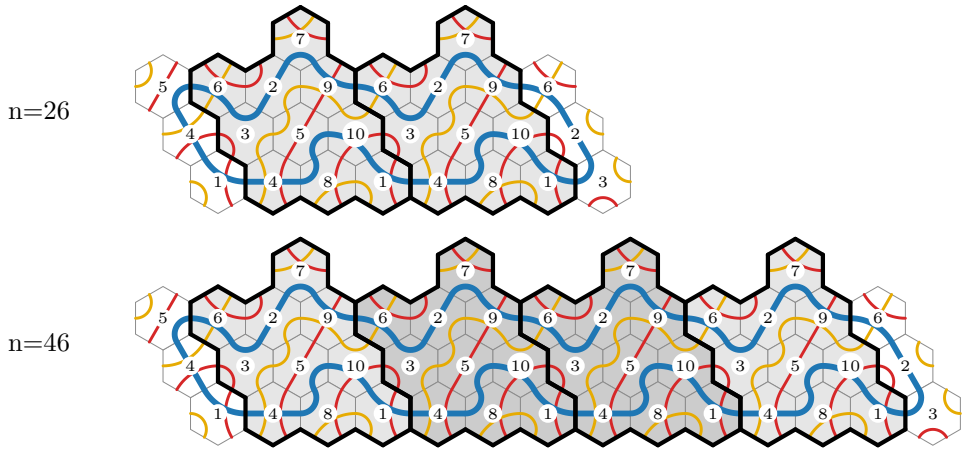



Figure 3: Top: the base $B_{6,\mathrm{b}}$ ($n_0 = 26$); the copies $G$ and $G + V$ are shaded and each outlined. Bottom: $B_2$, a solution for $n = 46$, obtained by inserting two copies of the gadget (darker shading). Every ten-tile copy of the gadget is outlined.

then they all lie in $[W-1, (k+1)W]$, which is contained in $U_2$ because $W \geq 1$. Consequently every adjacent pair of $B_k$ (together with the tiles on it) is an adjacent pair of $B$, of $\widehat{G}$ or of $B + kV$.

*(S1).* $|B_k| = n_0 + 10k$, and each inserted copy contributes one tile of every type by (G0). Since $m_t(n_0 + 10k) = m_t(n_0) + k$ for all $t$ (as $n_0 \geq 10$), (S1) holds.

*(S2) and the first part of (S3).* These are conditions on adjacent pairs, so by locality they follow from (S2), (S3) for $B$ and from (G1), (G2) for $\widehat{G}$. In particular $L_c(B_k)$ is a disjoint union of cycles.

*Single loop.* An edge of a loop graph between columns $W-1$ and $W$ joins cells of $[0, 2W) \times \mathbb{Z}$, where $B$ coincides with $\widehat{G}$; by (G2) there are exactly two such edges in $L_c(B)$, say $e_1 \in \Lambda_1$ and $e_2 \in \Lambda_2$. Removing them from the cycle $L_c(B)$ leaves two paths; since no other edge crosses between columns $W-1$ and $W$, one path $Q_L$ consists of the cells of $B$ with $x < W$ and the other path $Q_R$ of the cells with $x \geq W$. In $B_k$ the path $Q_L$ is unchanged (region $U_1$) and $Q_R$ is translated by $kV$ (region $U_3$). In region $U_2$, the components $\Lambda_1$ and $\Lambda_2$ of $L_c(\widehat{G})$ contain the segments from $e_1$ to $e_1 + kV$ and from $e_2 + kV$ to $e_2$, which lie in $[0, (k+2)W)$ and pass through every cell of the inserted copies. Hence $L_c(B_k)$ is the single cycle

$$\begin{aligned} Q_L \;\to\; e_1 \;\to\; (\text{segment of } \Lambda_1) \;\to\; e_1 + kV \;\to\; Q_R + kV \\ \to\; e_2 + kV \;\to\; (\text{segment of } \Lambda_2) \;\to\; e_2 \;\to\; Q_L, \end{aligned}$$

through all $n_0 + 10k$ cells, which proves (S3).

*(S4).* $B_k$ is connected by (S3), so by Lemma 2 it suffices to show $\chi(B_k) = 1$. Write $\kappa = E_W(\widehat{G}) - F_W(\widehat{G})$. Crossing pairs and triples at a cut use only the two columns next to the cut (Lemma 3), and near the cuts at $x = W$ and $x = (k+1)W$ the placement $B_k$ coincides with $\widehat{G}$; likewise $B$ coincides with $\widehat{G}$ near the cut at $x = W$. Applying Lemma 3 to $B$ at $x = W$ gives

$$\chi(B_{<W}) + \chi(B_{\geq W}) = \chi(B) + \kappa.$$

Applying it to $B_k$ at $x = W$ and at $x = (k+1)W$, and to the middle part $\widehat{G} \cap [W, (k+1)W)$ at the $k-1$ internal cuts $x = jW$, $2 \leq j \leq k$, gives

$$\chi(B_k) = \chi(B_{<W}) + \big(k\,\chi(G) - (k-1)\kappa\big) + \chi(B_{\geq W}) - 2\kappa = \chi(B) + k\big(\chi(G) - \kappa\big) = \chi(B) + k\,\chi_{\mathrm{per}}(G).$$

(For $k = 0$ the middle term is absent and the formula reduces to $\chi(B_0) = \chi(B)$.) By (G3) and $\chi(B) = 1$ we get $\chi(B_k) = 1$. □

## 6 Proof of the main theorem

**Theorem 8.** *For every $n \geq 3$ and every designated colour $c$ of $n$, the challenge $(n, c)$ has a solution. Moreover, $(n, \mathrm{b})$ has a solution for every $n \geq 3$ with $n \equiv 7 \pmod{10}$.*

*Proof.* Call $(r, c)$ *required* if $c$ is a designated colour of residue $r$, or $(r, c) = (7, \mathrm{b})$; there are thirteen required pairs. For each colour $c \in \{\mathrm{r}, \mathrm{y}, \mathrm{b}\}$ there is a gadget $G_c$ with $W = 3$, $H = 4$ (Figure 2), and for each required pair $(r, c)$ there is a base $B_{r,c}$ for $G_c$ with $n_0(r, c) \equiv r \pmod{10}$ (Table 2 and Appendix A). Both facts are verified by the checker described in Section 7. By Theorem 7, $(n, c)$ has a solution for every $n \geq n_0(n \bmod 10, c)$ with $n \equiv n_0 \pmod{10}$. Since $n_0(r, c) \leq 35$ for all required pairs except $n_0(7, \mathrm{b}) = 37$, this covers every $n \geq 36$. For $3 \leq n \leq 35$ explicit solutions of all required pairs are part of the certificates. □

| $(r, c)$ | 0r | 0y | 0b | 1y | 2y | 3y | 4r | 5r | 6b | 7r | 7b | 8b | 9y |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n_0$ | 30 | 30 | 30 | 31 | 32 | 33 | 34 | 35 | 26 | 27 | 37 | 28 | 29 |

Table 2: Sizes of the bases found by the search of Section 7.

## 7 Computation

All finite objects used in the proof were found with the constraint solver CP-SAT of Google OR-Tools [7] (version 9.8, C#) and then checked by a separate program. The search programs are not part of the proof; only the checker is.

**Fixed-board model.** For a set $C$ of candidate cells (a hexagonal disk of radius $R$ around the origin, or a part of the strip $S$), the model has a Boolean $u_p$ ("$p$ is used") for every cell $p \in C$, a Boolean arc variable for every ordered pair of adjacent cells, and a self-loop literal equal to $\neg u_p$. The global constraint `AddCircuit` then forces the used cells to form a single directed cycle, which will be the loop. Further Booleans select the entry and exit edge of the loop in each used cell, and the tile type and rotation; a table relates (type, rotation, entry, exit), so that the line of colour $c$ on the tile joins the entry and exit edges. The remaining constraints are $\sum_p u_p = n$, the tile counts $m_t(n)$, the matching condition (S2) on every adjacent pair of used cells, and the no-holes condition in the form $\chi(P) = 1$ (Lemma 2), written linearly with one product literal per adjacent pair and per triple of pairwise adjacent cells. Two redundant constraints help the solver. Measured in sixths of a full turn, the total turning of the loop is $+6$ or $-6$, depending on the direction of traversal; we fix it to $+6$ in the small instances, which only chooses the direction. The numbers of sharp, gentle and straight loop segments are determined by $n$, because the shape of the colour-$c$ line depends only on the tile type. As a symmetry break, the origin is used. The local rule of Lemma 4 is included as a further redundant constraint.

**Small certificates.** For $3 \leq n \leq 35$ and every required colour, we solved the model on disks of radius $R_{\min}, \ldots, R_{\min} + 3$, where $R_{\min}$ is the least radius whose disk has at least $n$ cells. Each instance took at most a few seconds with 12 worker threads. The same model solves larger challenges directly: red loops were found for $n = 50$, 60 and 70 in 36, 18 and 272 seconds. For $n = 80$ no solution was found within 300 seconds on radii 5 and 6. For comparison, [4] report 5549 seconds for $n = 50$ with an integer programming solver, on much older hardware. Earlier versions of the

model, written in ECLiPSe [2] with the Gecode-based `gfd` library and in MiniZinc [6], served as an independent cross-check of the model; the ECLiPSe version solved all challenges with $n \leq 40$.

**Gadgets.** A gadget is searched for directly on the cylinder $(\mathbb{Z}/W\mathbb{Z}) \times [0, H)$. Each used cell gets a strand label in $\{1, 2\}$ and a position variable. Position increases by one along every loop arc that does not cross the seam $x = W - 1 \to x = 0$. This forbids cycles that do not wrap around the cylinder. Each strand has exactly one arc across the seam, in the positive direction for strand 1 and in the negative direction for strand 2, and its total turning is 0. Every tile type occurs once, the two strands touch, and the cylindrical Euler characteristic is 0, which is (G3). For $W = 3$ and $H = 4$ the search found a gadget for each colour in less than a second.

**Bases.** With the gadget fixed in the columns $[0, 2W)$ (as $G$ and $G+V$), we solved the fixed-board model on the cells of $S$ with $-K \leq x < 2W+K$. The base size was $n_0 = 20+r+10j$ for $j = 0, 1, 2, 3$ in turn, and $K = n_0 - 19$; each value was tried with total turning $+6$ and $-6$. The thirteen bases of Table 2 were found in at most 16 seconds each; the smaller values of $n_0$ that were tried gave no base.

**Independent checker.** The certificates list every solution in loop order, one tuple (position, type, rotation, entry edge, exit edge) per tile. A checker of about 260 lines of JavaScript (Node.js, no dependencies), written independently of the search code, verifies the following:

1. conditions (S1)–(S4) for each of the 42 small certificates, where (S4) is tested by a flood fill of the empty cells and, separately, by $\chi = 1$;
2. conditions (G0)–(G3) for the three gadgets, directly on the cylinder;
3. the hypotheses of Definition 6 for the thirteen bases, and, as a sanity check of Theorem 7, conditions (S1)–(S4) for $B_k$ for $0 \leq k \leq 30$;
4. that every pair $(n, c)$ with $3 \leq n \leq 500$ and $c$ required is covered by a small certificate or by a base with $n_0 \leq n$, $n_0 \equiv n \pmod{10}$.

The check runs in a few seconds.

## 8 Discussion and open problems

**Roundness.** The solutions $B_k$ are strips of height 4 and grow only in one direction, so they fail the roundness condition required for official records [4], just like Fraser's 4-tiles' equation. Does every challenge have a *round* solution? A two-dimensional version of the insertion argument, with gadgets that can be inserted in two directions, would be one way to attack this question.

**Other colours.** The official sheet [9] states that the puzzles with fewer than ten tiles can be solved in only one colour. Under the rules (S1)–(S4) this is not the case for $n = 7$: blue loops exist for every $n \equiv 7 \pmod{10}$ (Theorem 8), including $n = 7$ itself. It would be interesting to determine all pairs $(n, c)$ that are solvable. Since the gadgets exist in all three colours, every colour is solvable for all large $n$ in a residue class as soon as a single base is found in that class.

**Smaller objects.** Our gadgets have width 3 and height 4, and the bases have $26 \leq n_0 \leq 37$. We did not try to minimise these sizes. A smaller $n_0$ would reduce the number of directly computed certificates. It is also natural to ask for the number of solutions of a challenge, or for the computational complexity of deciding whether a generalised Discovery challenge (with an arbitrary multiset of tiles) has a solution.

# A Certificate data

The machine-readable certificates are distributed as JSON files together with the checker:

- `certificates/small/N`*nn c*`.json`: the 42 small solutions;
- `certificates/gadgets/gadget_`*c*`.json`: the three gadgets;
- `certificates/bases/base_`*rc*`.json`: the thirteen bases.

Figure 4 and the following figures show all small solutions with $3 \leq n \leq 30$, one for each required colour; the loop is drawn as a thick line. Figures 7 and 8 then show the thirteen bases in order of increasing $n_0$. In each base the two copies $G$ and $G + V$ of the gadget are shaded and outlined. The new copies are inserted between them, as in Figure 3.

# References


[1] D. Baumeister and J. Rothe, Satisfiability parsimoniously reduces to the Tantrix™ rotation puzzle problem, *Fundamenta Informaticae* 91(1) (2009), 35–51. Preliminary version in MCU 2007, LNCS 4664, Springer.

[2] The ECLiPSe constraint programming system, `https://eclipseclp.org`.

[3] M. Holzer and W. Holzer, Tantrix™ rotation puzzles are intractable, *Discrete Applied Mathematics* 144(3) (2004), 345–358.

[4] F. Kino and Y. Uno, Solving Tantrix via integer programming, arXiv:1202.6438, 2012.

[5] F. Kino and Y. Uno, An integer programming approach to solving Tantrix on fixed boards, *Algorithms* 5(1) (2012), 158–175.

[6] N. Nethercote, P. J. Stuckey, R. Becket, S. Brand, G. J. Duck and G. Tack, MiniZinc: towards a standard CP modelling language, in *CP 2007*, LNCS 4741, Springer, 529–543.

[7] L. Perron and F. Didier, CP-SAT, Google OR-Tools, version 9.8, `https://developers.google.com/optimization`.

[8] J. Scherphuis, Tantrix, Jaap's Puzzle Page, `https://www.jaapsch.net/puzzles/tantrix.htm`.

[9] Tantrix puzzles, official Tantrix website, `https://www.tantrix.com/english/TantrixPuzzles.html`.

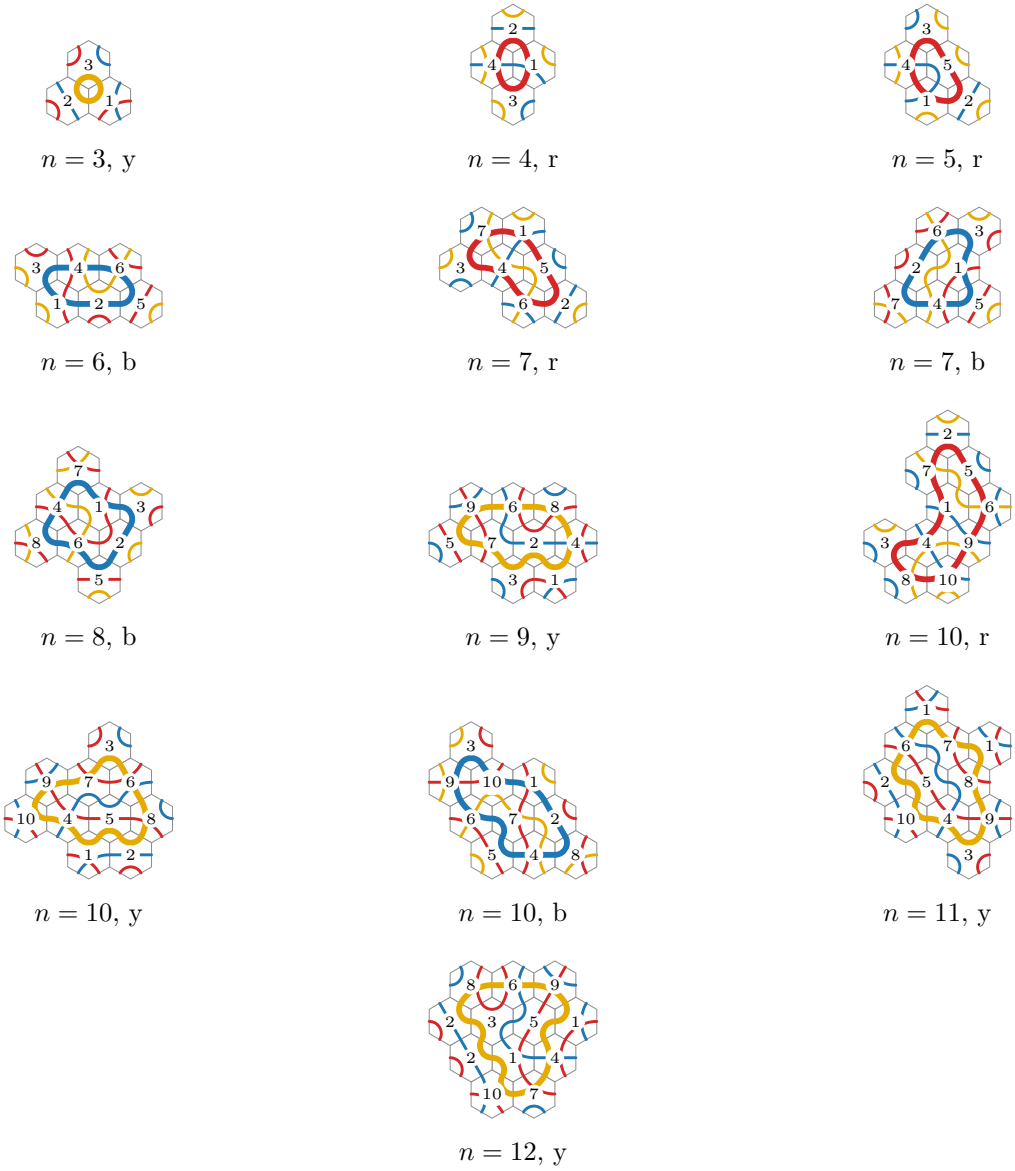
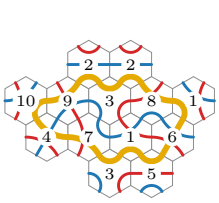
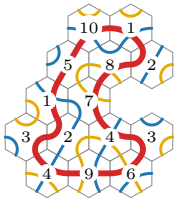


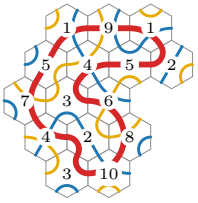

Figure 4: Solutions found directly by the CP-SAT model, $3 \leq n \leq 12$, with the loop colour (r, y, b).



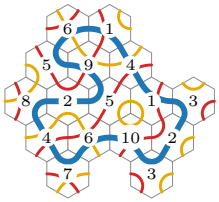
$n = 13$, y



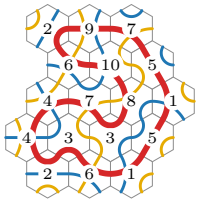
$n = 14$, r



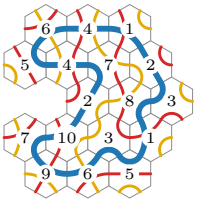
$n = 15$, r



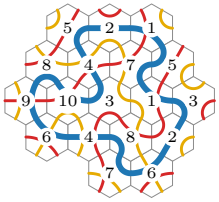
$n = 16$, b



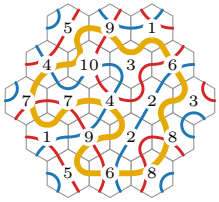
$n = 17$, r



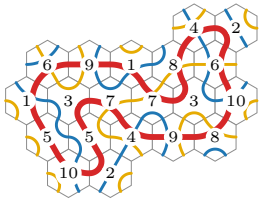
$n = 17$, b



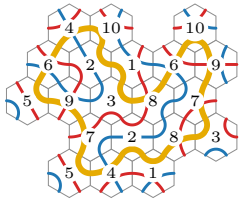
$n = 18$, b



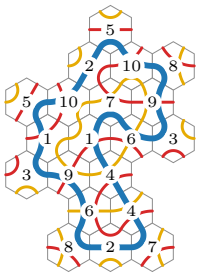
$n = 19$, y



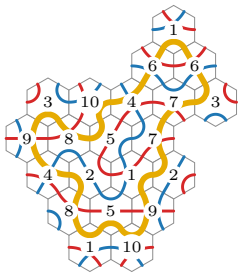
$n = 20$, r



$n = 20$, y



$n = 20$, b



$n = 21$, y

Figure 5: Solutions found directly by the CP-SAT model, $13 \leq n \leq 21$, with the loop colour (r, y, b).

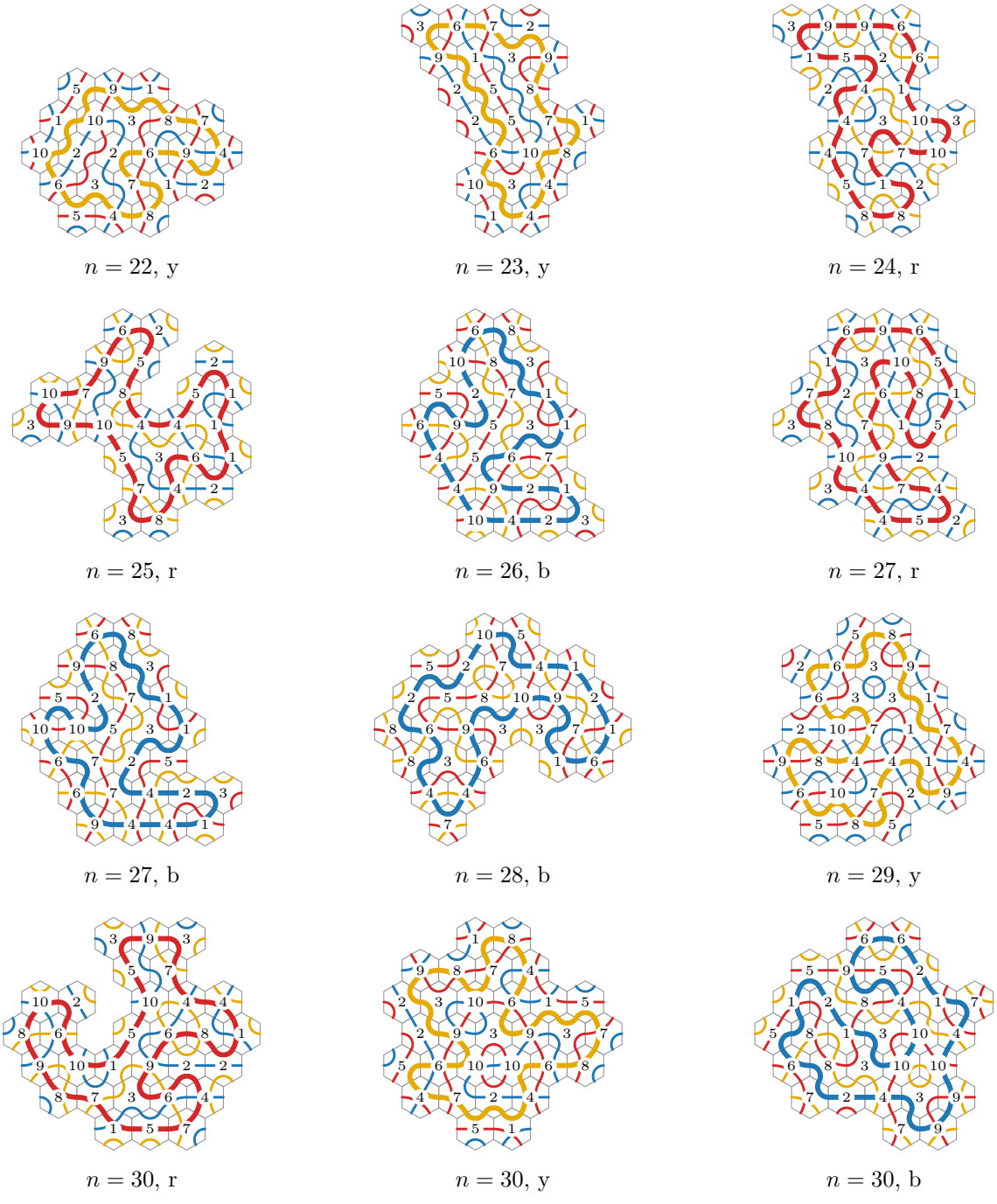



Figure 6: Solutions found directly by the CP-SAT model, $22 \leq n \leq 30$, with the loop colour (r, y, b).

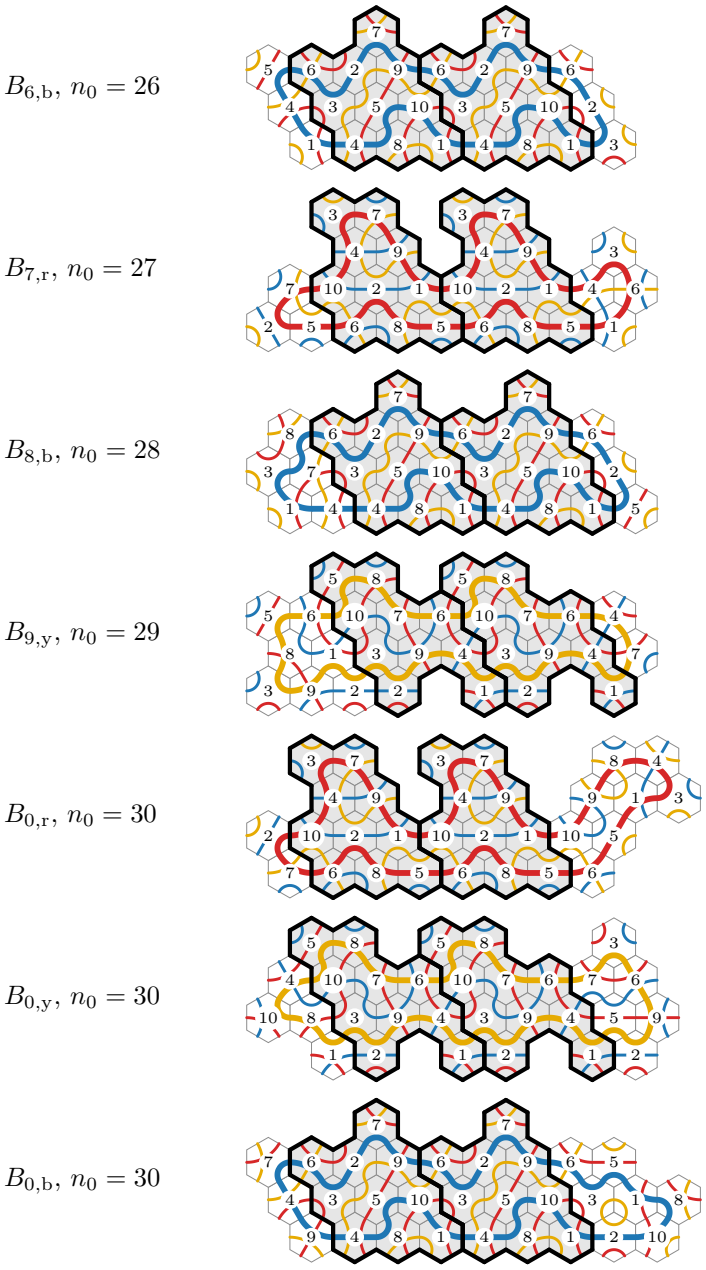



Figure 7: Bases (first part). The shaded and outlined ten-tile blocks are $G$ and $G + V$; the thick coloured line is the loop.

$B_{1,\mathrm{y}}$, $n_0 = 31$

$B_{2,\mathrm{y}}$, $n_0 = 32$

$B_{3,\mathrm{y}}$, $n_0 = 33$

$B_{4,\mathrm{r}}$, $n_0 = 34$

$B_{5,\mathrm{r}}$, $n_0 = 35$

$B_{7,\mathrm{b}}$, $n_0 = 37$

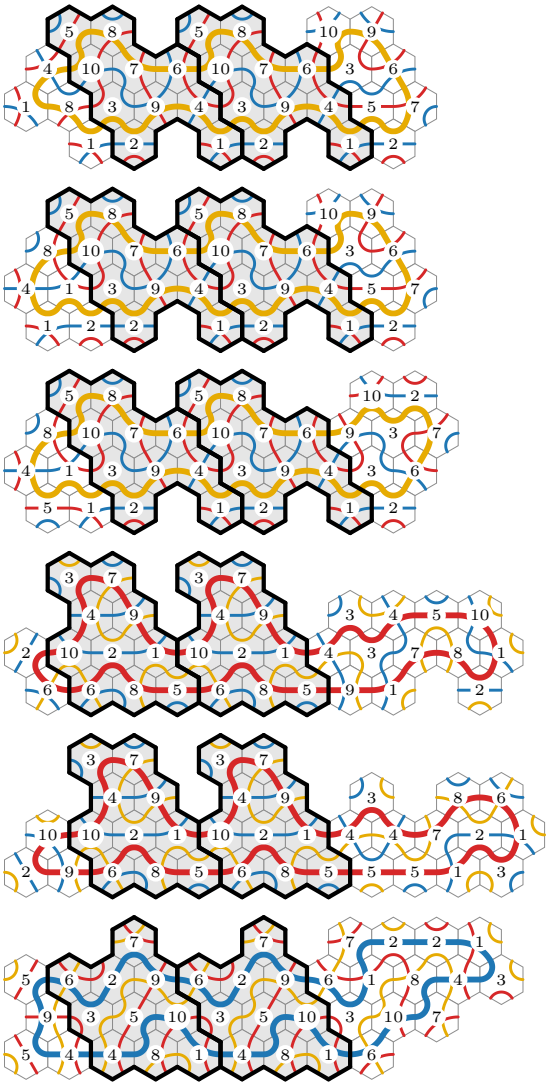


Figure 8: Bases (second part).